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A NOTE ON THE SERRE DIMENSION OF POLYNOMIAL RINGS

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A commutative noetherian ring R is said to have *Serre dimension* $\leq t$ if every projective R-module P of rank $\geq t+1$ has a unimodular element, i.e. $P = Rp \oplus P'$, for some $p \in P$.

Serre's classically well-known theorem asserts that the Serre dim $R \leq \dim R$. As pointed out by Plumstead in [P], the Eisenbud–Evans theorem shows that Serre dim $R \leq$ Generalised dim R. This enabled him to prove that the Serre dim $R[X] \leq \dim R$.

Recently, S.M. Bhatwadekar and Amit Roy (see [1]) extended Plumstead's arguments to show that Serre dim $R[X_1, \ldots, X_n] \leq \dim R$, thereby settling an old conjecture of Bass. In this note we present a different view point of the analysis 'in the fourth corner', based on a combination of a theorem of Swan and a variant of a theorem of Roitman.

The idea in [1] is to make the following proposition of Plumstead available in the present context;

Plumstead's Patching Proposition [3, Proposition 1 of Section II]. *Let B be any ring, $A = B[Z]$, and P a projective A-module. Let p' be an unimodular element of $\bar{P}$* ('bar' *meaning* 'modulo(Z)'). *Let s be an element of B and $T = 1 + sB$. Let p_1 (resp. p_2) be a unimodular element of P_s (resp. P_T) such that $\bar{p}_1 = p'_s$ (resp. $\bar{p}_2 = p'_T$). Set $N_1 = P_2/A_s p_1$ and $N_2 = p_T/A_T p_2$. Assume further that $(N_1)_T$ and $)N_2)_s$ are extended from B_{Ts}. Then p has a unimodular element p such that $\bar{p} = p'$.*

Let us give a brief sketch of how they arranged to utilize this proposition in [1]; so as to pinpoint how the 'fourth corner' ($B_{Ts}[Z]$ in the above proposition) looks.

Take a projective $R[X_1, \ldots, X_n]$-module P of rank $\geq d+1$, where $d = \dim R$. One asserts that it has a unimodular element lifting any given one 'modulo $(X_1, \ldots, X_n)$' by induction on the pair (d, n). The validity of Serre's conjecture and Plumstead's theorem enable us to start the induction and assume $d \geq 1$, $n > 1$.

Fix a unimodular element p_0 of $P/X_n P$. Choose $s \in R$ s.t. P_s is free; and select a unimodular element $p_s \equiv (p_0)_s$ (mod$(X_n P)$).

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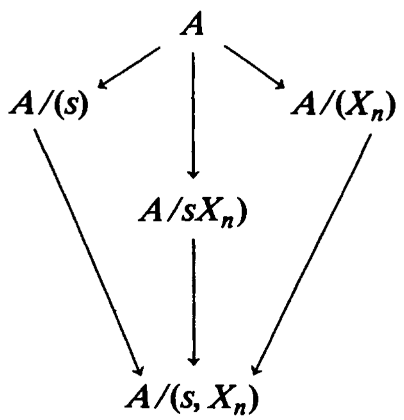
By induction, P/sX_nP has a unimodular element $\tilde{p}$. The Eisenbud-Evans theorem permits us to choose a lift p' of $\tilde{p}$ with ht $O(p')>d+1$. Let $g\in O(p')$ with $g\equiv 1$ (mod(s)).

A suitable change of variables $X_i\mapsto X_i'$(for $1<i<n-1$), $X_n\mapsto X_n$, forces $O(p')$ to have a monic polynomial f in X_n with its coefficients in $R[X_1',\dots,X_{n-1}']$ ($=B'$ say). Let $T=1+sB'$, $A=T^{-1}B'[X_n]$.

Since $h=\text{Resultant}(f,g)\in O(p')\cap T^{-1}B'$, and $h\equiv 1$ (mod(s)), p_T' is unimodular in P_T.

An easy diagram (see below) chase, aided by the double induction and that $V(sX_n)=V(s)\cup V(X_n)$, enables one to exhibit a unimodular element $p_T\in P_T$ lifting p_0.



Let $P_T=A_{p_T}\oplus N$. To make Plumstead's patching proposition available, we only need to show that N_s is extended from B'_{T_s}. The object of this note is to prove this. (In [1], they had to *arrange* p_T suitably to infer, by a deep theorem of Quillen-Suslin, that its cokernel became extended after inverting s.)

We observe that,

(1) N_s is stably free of rank $\geq d$, by the very choice of s.

(2) $A_s=B'_{Ts}[X_n]$ is a localisation of $(R_{(1+sR)s}[X_1',\dots,X_{n-1}'])[X_n]$ at the muliplicatively closed subset T of the base ring.

Note that $\dim R_{(1+sR)s}\leq d-1$.

We conclude that N_s is extended from B'_{Ts} by applying the following proposition.

Proposition 1. *Let* $B=R[X_1,\dots,X_m]$, *with* $\dim R=d$. *Let S be a multiplicatively closed subset of B. Then any stably extended $S^{-1}B[Z]$-projective module of rank $\geq d+1$ is extended from $S^{-1}B$.*

Proof. By Quillen's localisation principle [4, 4.15.4] we need show that every stably free $(S^{-1}B)[Z]$-module is free, for every $\mathfrak{p}\in\operatorname{Spec} S^{-1}B$.

We may safely assume that the projective module is defined by a unimodular row $w(Z)=(w_1(Z),\dots,w_{t+1}(Z))$, $t\geq d$.

Since $\mathrm{Sl}_{t+1}(R_\mathfrak{p})=\mathrm{E}_{t+1}(R_\mathfrak{p})$, and elementary matrices can be lifted, we may assume $w(0)=(1,0,\dots,0)$.

Let $w(Z)=(1+Zw_1'(Z), Zw_2'(Z),\dots,Zw_{t+1}'(Z))$, for some $w_i'(Z)\in R[Z]$, $1\le i\le t+1$.

Let $w_i'(Z)\in R_s[Z]$, for some $s\notin \mathfrak{p}$. Put $w_i'(Z)=w_i''(Z)/s^m$, for some $w_i''(Z)\in R[Z]$, $m\ge 0$.

Let $Z'=Z/s^{m+1}$. Then

$$w(Z)=\left(1+sZ\frac{w_1''(Z)}{s^{m+1}}, sZ\frac{w_2''(Z)}{s^{m+1}},\dots,sZ\frac{w_{t+1}''(Z)}{s^{m+1}}\right)$$

$$=(1+sZ'w_1''(s^{m+1}Z'), sZ'w_2''(s^{m+1}Z'),\dots,sZ'w_{t+1}''(s^{m+1}Z')).$$

The last row (call it $w'(Z')$) is actually unimodular when regarded as a row over $B[Z']$. This is because it has an element $1+sZ'w_1''(s^{m+1}Z')\equiv 1 \pmod{(s)}$, and is unimodular after inverting s.

By [6, Theorem 1.1], there exists a $\sigma'\in \mathrm{Gl}_{t+1}(R[Z'])$ s.t. $\sigma'w'=(1,0,\dots,0)=w'(0)$.

Changing back to our original variables, we have a $\sigma\in\mathrm{Gl}_{t+1}(R_s[Z])$ with $\sigma w=(1,0,\dots,0)$.

Remark 1. One may imitate the above proof to show the following variant of a theorem of Roitman [5, Proposition 2].

Proposition 2. *Let S be a multiplicatively closed subset of R. Assume that every stably free projective $R[Z]$-module of rank $\ge t$ is extended from R. Then every stably extended projective $S^{-1}R[Z]$-module of rank $\ge t$ is extended from $S^{-1}R$.*

Remark 2. The above analysis in the fourth corner was worked out when $n>1$. A corresponding approach may be presented when $n=1$. The ring in the fourth corner is $B_{Ts}[Z]$, where $T=1+sR$, s being chosen to that P_s is free. One then has two unimodular vectors u,w in P_{sT} 'sitting above' a given unimodular vector in $P_{sT}/(Z)$. It suffices to find a $\alpha\in G_{d+1}(R_{sT}[Z])$ with $\alpha u=w$.

If $\dim R=1$, then $(R_{sT})_{\mathrm{red}}$ is a product of fields, and this is obvious. Let $d=\dim R>1$.

Then $\forall\mathfrak{p}\in\operatorname{Spec} R_{sT}$, $(R_{sT})_{\mathfrak{p}}[Z]$ has generalised dimension $\le d-1$ by [3, Section 1, Example 4].

Consequently, by the Eisenbud–Evan's theorem, there exists $\delta_{\mathfrak{p}}$, $Z_{\mathfrak{p}}\in G_{d+1}((R_{sT})_{\mathfrak{p}}[Z]$ s.t. $\sigma_{\mathfrak{p}}u=u(0)$, $Z_{\mathfrak{p}}w=w(0)$.

A cute theorem of Vasertein (see [2, p. 87, Theorem 25]) now says that there exists $\sigma,\tau\in G_{d+1}(R_{sT}[Z])$ s.t. $\sigma u=u(0)$, and $\tau(w)=w(0)$.

Since $u(0)=w(0)$, $\tau^{-1}\sigma$ maps u to w.

References

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